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Fractional strings in (p, q) 5-brane and quiver Matrix string theory

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Abstract: We study the (p, q)5-brane dynamics from the viewpoint of Matrix string theory in the T-dualized ALE background. The most remarkable feature in the (p, q)5-brane is the existence of "fractional string", which appears as the instanton of 5-brane gauge theory. We approach to the physical aspects of fractional string by means of the two types of Matrix string probes: One of which is that given in Witten (*New "gauge" theories in six dimensions*, *JHEP* **01** (1998) 001 [hep-th/9710065]). As the second probe we present the Matrix string theory describing the fractional string itself. We calculate the moduli space metrics in the respective cases and argue on the specific behaviors of fractional string. Especially, we show that the "joining" process of fractional strings can be realized as the transition from the Coulomb branch to the Higgs branch of the fractional string probe. In this argument, we emphasize the importance of some monodromies related with the θ-angle of the 5-brane gauge theory.

Keywords: D-branes, M(atrix) Theories, Brane Dynamics in Gauge Theories.

Contents

1. Introduction

Analyses of the brane dynamics have been playing a central role in the studies on string duality and M-theory. Among others, the studies of 5-brane have a primary importance and gives rich products. This is because the 5-brane is the magnetic dual of string (and also the dual of the M-theory membrane), and from the viewpoint of Matrix string theory [1], the degrees of freedom of 5-brane are naturally incorporated into the theory by considering some matter fields (hypermultiplets) [2, 3].

In the limit when the gravitational interaction to the bulk theory decouples, the 5-brane dynamics leads to the 6-dimensional "new" quantum theory [4]. The most remarkable feature of this 6-dimensional theory is the existence of a non-local excitation (non-critical string). In the set up from the IIB 5-branes, whose low energy effective theory is a 6-dimensional gauge theory, this non-critical string appears as the instanton. In the system of parallel NS5-branes (D5-branes) with the vanishing IIB θ-angle, this naturally identified with the fundamental string (D-string) trapped inside their world volumes [4]. However, as is emphasized in [5], in the (p,q) 5-brane [1] cases (or equivalently, NS5s or D5s with some rational θ-angle) we face different circumstances. One might imagine the instanton string can be constructed by the simple $SL(2,\mathbf{Z})$-duality transformation from the fundamental string (in the NS5 theory, and, of course,

[1] We use the term "(p,q) 5-brane" as the meaning of the bound state of p D5s and q NS5s, according to the convention in [6].

D-string for D5). But this is *not* correct. It is known that the instanton string in the (p,q) 5-brane has the following tension [5, 7]

$$T_{p,q}^{\rm instanton} = \frac{{\rm Im}\tau}{|a+b\tau|}T\,, \qquad ((p,q)=r(a,b)\,,\quad a,b:\ \text{coprime}) \tag{1.1}$$

where T denotes the tension of the fundamental string and τ is the IIB complex coupling $\tau = \frac{\theta_{\rm IIB}}{2\pi} + i\frac{1}{g_s}$. For generic values of τ, this does not coincide with *any* string tension of the $SL(2,\mathbf{Z})$-multiplet of BPS strings (so-called "(r,s)-string"): $T_{r,s} = |r - s\tau|T$. Especially, under the decoupling limit $\tau \longrightarrow i\infty$, we have $T_{p,q}^{\rm instanton} = T/b$, which is b-times smaller than that of the fundamental string. So it is plausible to call it as the "fractional string".

We intend in this paper to study the property of fractional string from the stand point of Matrix theory [8]. To this aim, it is a natural set up to take the fractional string as the Matrix string probe. But this is not so easy, because, as we just commented, the fractional string is not a D-brane (nor the object obtained from a D-brane by U-duality) in the usual sense. Therefore we shall take the T-dualized framework - M-theory compactified on some "twisted" orbifold [6], and use the technique of the quiver Matrix theory introduced in [9, 10].

In section 2 we shall begin by reviewing the M-theory picture corresponding to the (p,q) 5-brane and its Matrix theory realization given in [6]. We calculate the moduli space metric and discuss how we can observe the exciteations of fractional string in this framework.

In section 3 we construct the quiver Matrix theory defined on the fractional string probe. This is a more direct approach to the dynamics of this object than that of [6]. We will show some existence of monodromies related with the θ-angle of the 6-dimensional gauge theory, which will play an important role in our analysis of the moduli space. We find out a different structure in its moduli space from that of the usual D-brane probe. This observation of ours will clarify the peculiar behaviors of fractional string.

Section 4 is devoted to the discussions and some comments.

2. Witten's matrix string probing (p,q) 5-brane

2.1. M-theory picture of the (p,q) 5-brane

We shall begin from the T-dualized construction of (p,q) 5-brane theory introduced in [6]. The claim is that, under the T-duality along one of the transversal direction, (p,q) 5-brane in IIB string theory is described by M-theory compactified on the "twisted" orbifold

$$X_{p,q} \stackrel{\rm def}{=} (\mathbf{C}^2 \times S^1)/\mathbf{Z}_q\,, \tag{2.1}$$

where the $\mathbf{Z}_q$-action is defined by

$$\begin{cases} z_1 \longrightarrow \omega z_1 \\ z_2 \longrightarrow \omega^{-1} z_2 \\ u \longrightarrow \omega^{-p} u \end{cases} \qquad (\omega \stackrel{\text{def}}{=} e^{2\pi i/q})\,. \tag{2.2}$$

(z_1, z_2) is the coordinate of $\mathbf{C}^2$ and u denotes that of S^1. Let us assume that $\gcd(p,q) = r$, and set $(p,q) = r(a,b)$. In this case $X_{p,q}$ has an A_{r-1}-singularity, which corresponds to the $U(r)$-gauge symmetry on the 5-brane.

Roughly speaking, the above statement can be explained as follows: Consider the NS5 in IIB string theory wrapping around the $0,\ldots,5$-th directions. If we take the T-duality along one of the transversal direction, say, the 6-th direction, we obtain IIA string theory compactified on the Taub-NUT space with the S^1-fibration along the dual 6-direction (we shall call it the "TN-direction");

$$\text{NS5} \stackrel{T^6}{\longleftrightarrow} IIA/TN \cong M/(TN \times S^1) \quad \text{(The TN-direction is the 6-th axis.)} \tag{2.3}$$

Similarly, D5 is mapped to the D6 by T-duality, and D6 is interpreted as M/TN with the TN-direction is the 11-th axis;

$$\text{D5} \stackrel{T^6}{\longleftrightarrow} D6 \cong M/(TN \times S^1) \quad \text{(The TN-direction is the 11-th axis.)} \tag{2.4}$$

In this way, we can expect the bound state of p D5s and q NS5s corresponds to the $M/(TN \times S^1)$ with the TN-direction $a\mathbf{e}_{11} + b\mathbf{e}_6$. ($S^1$ is also wrapping around $c\mathbf{e}_{11} + d\mathbf{e}_6$, where c, d is the integers uniquely determined from a, b, so that $\begin{pmatrix} a & c \\ b & d \end{pmatrix} \in SL(2,\mathbf{Z})$.) $X_{p,q}$ roughly has this $TN \times S^1$ structure (under the limit when the TN-circle decompactifies) away from the singular point ($z_1 = z_2 = 0$).

However, the role of singularity is essential for our discussion. At the singular point, the TN-circle shrinks (the vanishing cycle) and we encounter an "exotic" circle which has a fractional radius [6]. This is because, by the identification of the $\mathbf{Z}_q$-action (2.2), the points $(z_1, z_2, u) = (0, 0, u)$ are identified with $(0, 0, \omega^{-p} u)$, which produces a fractional circle. It is in fact the origin of fractional string. Namely, the M2-brane wrapping arround this fractional circle appears as a string possessing the fractional tension.

2.2. Witten's Matrix string and screwing procedure

Considering the compactification over another S^1 (say, the 5-th direction) : $M/(X_{p,q} \times S^1)$, one can introduce the DVV's Matrix string theory [1] which has the KK momentum along this extra S^1. By means of the U-duality ($5 \leftrightarrow 11$-flip), we can reduce this system to the D0-probes in $IIA/X_{p,q}$. Now, the $\mathbf{T}^2$-fiber of $X_{p,q}$ is wrapping around the 5,6-th directions. (TN-direction becomes $a\mathbf{e}_5 + b\mathbf{e}_6$.) It is the starting point of the Matrix theory realization in [6].

Let us trace the construction of this Matrix theory. The bosonic part of the action of D0-branes in flat space is

$$S = T_{D0} \int dt \ \mathrm{Tr} \left(\frac{1}{2} \sum_{I=1}^{9} (D_0 X^I)^2 + \frac{T^2}{4} \sum_{I,J=1}^{9} [X^I, X^J]^2 \right) \tag{2.5}$$

where $T_{D0} = 1/g_s^A l_s$ is the mass of D0-brane and $T = 1/2\pi l_s^2$ is the tension of the fundamental string. We decompose nine-dimensional transverse space into $\mathbf{R}^4 \times \mathbf{R} \times \mathbf{C}^2$,

$$\begin{cases} \mathbf{R}^4 : X^i \quad (i = 1,2,3,4) \\ \mathbf{R} \ \ : X^5 \\ \mathbf{C}^2 : X^a \quad (a = 6,7,8,9) \\ \quad Q = X^6 + iX^7 , \ \tilde{Q} = X^8 + iX^9 \end{cases} \tag{2.6}$$

First we remark that $X_{p,q} \equiv (\mathbf{C}^2 \times S^1)/\mathbf{Z}_q \cong (\mathbf{R} \times \mathbf{C}^2)/\Gamma$, where the action of the Abelian group $\Gamma = \{\alpha, \beta\}$ on $\mathbf{R} \times \mathbf{C}^2$ is given by,

$$\begin{array}{llll} \alpha : Q \to \omega Q, & \tilde{Q} \to \omega^{-1} \tilde{Q} , & X^5 \to X^5 - 2\pi R_5 p/q \\ \beta : Q \to Q, & \tilde{Q} \to \tilde{Q}, & X^5 \to X^5 + 2\pi R_5 \, . \end{array} \tag{2.7}$$

So, our task is to construct the Matrix theory with the Γ-invariance imposed. The Chan-Paton factor is labeled by the element of Γ, and Matrix element is expressed as $\langle g|X^I|g'\rangle$ $(g, g' \in \Gamma)$. We can represent an element of Γ in terms of the generator, as $\alpha^s \beta^m = (s,m)$, where $s \in \mathbf{Z}_q$ and $m \in \mathbf{Z}$. The transformation law of X^I under Γ is expressed as

$$\begin{aligned} &\langle s+t, m+n|X^i|s'+t, m'+n\rangle = \langle s,m|X^i|s',m'\rangle \\ &\langle s+t, m+n|X^5|s'+t, m'+n\rangle = \langle s,m|X^5|s',m'\rangle + 2\pi R_5 (n - \frac{p}{q} t) \langle s,m|s',m'\rangle \\ &\langle s+t, m+n|Q|s'+t, m'+n\rangle = \omega^t \langle s,m|Q|s',m'\rangle \\ &\langle s+t, m+n|\tilde{Q}|s'+t, m'+n\rangle = \omega^{-t} \langle s,m|\tilde{Q}|s',m'\rangle \end{aligned} \tag{2.8}$$

where $\langle s,m|s',m'\rangle = \delta_{s,s'}\delta_{m,m'}$. By these relations (2.8), we can define the "reduced Matrix element" $\langle g|X\rangle$ of Matrix X by

$$\langle s,m|X\rangle := \langle s,m|X|0,0\rangle \, , \tag{2.9}$$

in the same way as [11]. It is useful to make further the Fourier transformation with respect to Γ. Namely, one can transfer the basis of Chan-Paton Hilbert space from that for Γ to that for the irreducible representations of Γ (we denote it as Γ^*), which is labeled by $R = (k, \theta) \in \mathbf{Z}_q \times \tilde{S}^1$. ($\tilde{S}^1$ is the dual circle along the 5-th direction.) The transformation coefficient $\langle R|g\rangle$ $(g \in \Gamma, R \in \Gamma^*)$ is nothing but the chracter of the irrep. R:

$$\langle R|g\rangle = \langle k,\theta|s,m\rangle = \omega^{ks} e^{i\theta(m - \frac{p}{q}s)} \, , \tag{2.10}$$

and one can immediately notice the following properties:

$$\begin{aligned}
\langle k,\theta|s+q,m+p\rangle &= \langle k,\theta|s,m\rangle \\
\langle k,\theta+2\pi|s,m\rangle &= \langle k-p,\theta|s,m\rangle\,.
\end{aligned} \tag{2.11}$$

Now, we can write down the Matrix element with respect to the Γ^*-basis:

$$\begin{aligned}
\langle k,\theta|X^i|k',\theta'\rangle &= 2\pi q\delta_{k,k'}\delta(\theta-\theta')X^i_k(\theta) \\
\langle k,\theta|X^5|k',\theta'\rangle &= \left(-2\pi R_5 i\frac{d}{d\theta}+X^5_k(\theta)\right)2\pi q\delta_{k,k'}\delta(\theta-\theta') \\
\langle k,\theta|Q|k',\theta'\rangle &= 2\pi q\delta_{k,k'-1}\delta(\theta-\theta')Q_{k,k+1}(\theta) \\
\langle k,\theta|\tilde{Q}|k',\theta'\rangle &= 2\pi q\delta_{k,k'+1}\delta(\theta-\theta')\tilde{Q}_{k,k-1}(\theta)
\end{aligned} \tag{2.12}$$

where we have introduced the Matrix variables expressing the reduced Matrix element for Γ^*-basis;

$$\begin{aligned}
X^i_k(\theta) &= \langle k,\theta|X^i\rangle \\
X^5_k(\theta) &= \langle k,\theta|X^5\rangle \\
Q_{k,k+1}(\theta) &= \langle k,\theta|Q\rangle \\
\tilde{Q}_{k,k-1}(\theta) &= \langle k,\theta|\tilde{Q}\rangle\,.
\end{aligned} \tag{2.13}$$

From (2.11), these variables should satisfy the following monodromy (so-called the "clock-shift");

$$\begin{aligned}
X^i_k(\theta+2\pi) &= X^i_{k-p}(\theta) \\
X^5_k(\theta+2\pi) &= X^5_{k-p}(\theta) \\
Q_{k,k+1}(\theta+2\pi) &= Q_{k-p,k+1-p}(\theta) \\
\tilde{Q}_{k,k-1}(\theta+2\pi) &= \tilde{Q}_{k-p,k-1-p}(\theta)
\end{aligned} \tag{2.14}$$

Rescaling further $A_1 = TX^5$ and

$$x^1 = \tilde{R}_5\theta = \frac{l_s^2}{R_5}\theta\,, \tag{2.15}$$

we obtain the expression of the original D0-brane action written in the Fourier transformed variables:

$$S = T_{D1}\frac{1}{q}\sum_{k=0}^{q-1}\int dt\int_0^{2\pi\tilde{R}_5}dx^1\mathrm{Tr}_N L_k\,, \tag{2.16}$$

where $T_{D1} = T/g_s^B = TR_5/g_s^A l_s$ is the tension of D1-brane and Tr_N is the trace over $U(N)$ indices. L_k is given by

$$\begin{aligned}
L_k = &-\frac{1}{4T^2}F_{\mu\nu,k}F_k^{\mu\nu} - \frac{1}{2}\Big\{(D_\mu X^i_k)^2 + |D_\mu Q_{k,k+1}|^2 + |D_\mu\tilde{Q}_{k,k-1}|^2\Big\} \\
&-\frac{T^2}{2}\left\{-\frac{1}{2}[X^i_k,X^j_k]^2 + \Big|[X^i,Q]_{k,k+1}\Big|^2 + \Big|[X^i,\tilde{Q}]_{k,k-1}\Big|^2 + \Big|\mu_k^{\mathbf{C}}\Big|^2 + \Big(\mu_k^{\mathbf{R}}\Big)^2\right\}\,.
\end{aligned} \tag{2.17}$$

Here we have defined

$$
\begin{aligned}
D_\mu Q_{k,k+1} &= \partial_\mu Q_{k,k+1} + i(A_{\mu,k}Q_{k,k+1} - Q_{k,k+1}A_{\mu,k+1}) \\
[X^i, Q]_{k,k+1} &= X^i_k Q_{k,k+1} - Q_{k,k+1}X^i_{k+1} \\
\mu^{\mathbf{C}}_k &= [Q, \tilde{Q}]_k \\
&= Q_{k,k+1}\tilde{Q}_{k+1,k} - \tilde{Q}_{k,k-1}Q_{k-1,k} \\
2\mu^{\mathbf{R}}_k &= [Q, Q^\dagger]_k + [\tilde{Q}, \tilde{Q}^\dagger]_k \\
&= Q_{k,k+1}Q^\dagger_{k,k+1} - Q^\dagger_{k-1,k}Q_{k-1,k} + \tilde{Q}_{k,k-1}\tilde{Q}^\dagger_{k,k-1} - \tilde{Q}^\dagger_{k+1,k}\tilde{Q}_{k+1,k}\,. \qquad (2.18)
\end{aligned}
$$

The action (2.16) gives the well-known form of the $U(N)$ A_{q-1} quiver gauge theory on $\mathbf{R}\times\mathbf{S}^1_{\tilde{R}_5}$, but *with the monodromy* (2.14) [6].

Remember the assumption $(p,q) = r(a,b)$ and $\gcd(a,b) = 1$. Taking account of the monodromy (2.14), one can reformulate this Matrix theory (2.16) on the b-times covering circle $\mathbf{S}^1_{b\tilde{R}_5}$. This "screwing procedure" is phrased as follows: For the set of functions $\{f_k(\theta)\}_{k=0}^{q-1}$ with the relation $f_k(\theta + 2\pi) = f_{k-p}(\theta)$, we define

$$
\begin{aligned}
f_{\hat{k}}(\theta) = f_{k-pl}(\theta - 2\pi l) \qquad & 2\pi l \le \theta \le 2\pi(l+1) \\
& (l = 0, \ldots, b-1 \quad \hat{k} = 0, \ldots, r-1)\,. \qquad (2.19)
\end{aligned}
$$

Then $f_{\hat{k}}(\theta)$ has the period $2\pi b$; $f_{\hat{k}}(\theta + 2\pi b) = f_{\hat{k}}(\theta)$. By this procedure, we can reduce the Matrix theory (2.16) to an A_{r-1} quiver theory on the "long string" $\mathbf{S}^1_{b\tilde{R}_5}$;

$$
S = \frac{T_{D1}}{b}\frac{1}{r}\sum_{\hat{k}=0}^{r-1}\int dt \int_0^{2\pi b\tilde{R}_5} dx^1 \mathrm{Tr}_N L_{\hat{k}}\,. \qquad (2.20)
$$

The fact that the theory has reduced to the A_{r-1} quiver is not surprising, because we have already known that $X_{p,q}$ actually possesses an A_{r-1}-singularity. The Higgs branch moduli space is known [15] to have the structure $Sym^N(ALE(A_{r-1}))$, which merely corresponds to the picture that free D-particles move in the ALE space with A_{r-1}-singularity.

However, the Coulomb branch moduli, which describes the 6-dimensional dynamics decoupled from the bulk, is non-trivial and more interesting.

2.3. Coulomb branch moduli space and fractional string

Recall that the Matrix theory (2.20) is defined on $\mathbf{R}\times\mathbf{S}^1_{b\tilde{R}_5} \equiv \mathbf{R}\times\mathbf{S}^1_{bl_s^2/R_5}$. The Coulomb branch $\mathcal{M}_V$ is hence parametrized by $\phi^i_m \in \left(\mathbf{R}^4 \times \mathbf{S}^1_{R_5/b}\right)^N$ (the Cartan components of 2-dimensional $N = (4,4)$ vectormultiplet and the Wilson line).[2] Here the superscript $i(= 0, \ldots, r-1)$ labels the nodes of quiver, and $m(= 1, \ldots, N)$ denotes the color index of $U(N)$.

[2]Precisely speaking, we should define $\mathcal{M}_V$ as the quotient space by the isometry $\phi^i_m \to \phi^i_m + \phi^{(0)}$.

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At tree level, the metric is diagonal: $ds^{2\ (0)} = \frac{TR_5}{qg_s^A l_s}\delta_{ij}\delta^{mn}\, d\phi^i_m d\phi^j_n$.

Quantum correction for the metric of the Coulomb branch comes from the integration of the massive fields on this branch, which are the off-diagonal component of the vector multiplet $\vec{X}^i_{mn}$ $(m \neq n)$ and the hypermultiplet $(Q^{i,i+1}_{mn}, \tilde{Q}^{i+1,i}_{nm})$. The masses of these fields are given by ϕ^{ii}_{mn} and $\phi^{i,i+1}_{mn}$, respectively, where we set $\phi^{ij}_{mn} \stackrel{\text{def}}{=} \phi^i_m - \phi^j_n$.

It is well-known that the correction exists only in the one-loop level by SUSY cancellation (and in this case, we have no instanton correction). Evaluating all the contributions of one-loop Feynman diagrams associated with the above massive modes, we obtain the following result;

$$\begin{aligned}
g_{ii}^{mm\ (1)} &= -2\sum_{n\neq m} G_1(\phi^{ii}_{mn}; bl_s^2/R_5) + \sum_{n,j\neq i}\hat{a}_{ij}G_1(\phi^{ij}_{mn}; bl_s^2/R_5) \\
g_{ii}^{mn\ (1)} &= 2G_1(\phi^{ii}_{mn}; bl_s^2/R_5) \qquad (m\neq n) \\
g_{ij}^{mn\ (1)} &= -\hat{a}_{ij}G_1(\phi^{ij}_{mn}; bl_s^2/R_5)\,, \qquad (i\neq j)
\end{aligned} \tag{2.21}$$

where the function $G_1(\phi^{ii}_{mn}; bl_s^2/R_5)$ is written in terms of the modified Bessel function K_1;

$$G_1(\phi, R) = \frac{1}{2|\vec{X}|^2}\left\{1 + 2\sum_{k=1}^{\infty} m_k|\vec{X}|K_1\Big(m_k|\vec{X}|\Big)\cos(m_k y)\right\}, \tag{2.22}$$

where $\phi = (\vec{X}, y) \in \mathbf{R}^4 \times \mathbf{S}^1_{R_5/b}$ (see Appendix for the detail). This is nothing but the Green function on $\mathbf{R}^4 \times \mathbf{S}^1_{R_5/b}$. $\hat{a}_{ij}$ stands for the adjacency Matrix of the A_{r-1} affine Dynkin diagram. The above one-loop metric (2.21) can be written in a simple form,

$$ds^{2\ (1)} = \frac{1}{2}\sum_{m,n,i,j} a^{mn}_{ij} G_1(\phi^{ij}_{mn}; bl_s^2/R_5)\Big(d\phi^{ij}_{mn}\Big)^2\,, \tag{2.23}$$

with $a^{mn}_{ij} = 2\delta_{mn}\delta_{ij} - \hat{C}_{ij}$, where $\hat{C}_{ij}$ is the A_{r-1} affine Cartan Matrix.

The behavior of this metric is characterized by the Green function $G_1(\phi; bl_s^2/R_5)$. It is worth remarking that it reflects the effects of the fractional string excitations. In fact, let us give a naive estimation of metric and compare it with above result. Since $X_{p,q}$ is $\mathbf{T}^2$-fibered, if one consider the D0-probe theory under the $X_{p,q}$-background, the loop calculation will need the summation of winding modes around this $\mathbf{T}^2$. From the construction, this $\mathbf{T}^2$ may have a non-trivial moduli. But, in the limit that the TN-circle decompactifies $R_6 \to \infty$, this $\mathbf{T}^2$ becomes a simple rectangular torus, and there survive only the winding modes $\sim nTR_5$. On the other hand, we know $X_{p,q}$ has an A_{r-1}-singularity, which will imply an A_{r-1} quiver gauge theory on D0-brane. The A_{r-1} quiver theory with the summation of these winding modes will give the Coulomb branch metric of the same form as (2.21), *but with* $G_1(\phi; l_s^2/R_5)$ *instead of* $G_1(\phi; bl_s^2/R_5)$. Of course, this is not the correct result, since this naive estimation forgets the fractional windings $\sim n\frac{TR_5}{b}$ inside the singular surface. The summation of these modes will give the correct answer (2.21).

Note the fact that the Witten's Matrix string becomes an A_{r-1} quiver *after taking the screwing procedure*, and in this procedure, it automatically includes the "maximally twisted sector". This is no other than the excitations of fractional string.

Although this consideration seems satisfactory, it feels somewhat indirect for our purpose. In the next section, we try to perform a more direct approach to the physics of fractional string, that is, the Matrix string theory on the fractional string probe.

3. Fractional string probe

In this section, we shall try to construct the Matrix theory describing the fractional string. As we already mentioned, the fractional string is realized as the M2-brane wrapped around the fractional circle $\sim \mathbf{e}_{11}/b$ at the singularity of $(\mathbf{S}^1 \times \mathbf{C}^2)/\mathbf{Z}_q$. After the $X^{11} - X^5$ flip, the instanton string is represented by the $D2$-brane wrapped around $\sim \mathbf{e}_5/b$. To obtain the desired theory, we consider the $D2$-brane on $(\mathbf{R} \times \mathbf{C}^2)/\Gamma$ whose world-volume is extended to X^0, X^1, X^5-direction. We identify one spacial coordinate σ of the $D2$-brane world-volume and target space coordinate X^5. Thus, the theory on the fractional string is given by the $(2+1)$-dimensional $U(q)$ supersymmetric Yang-Mills theory projected by the following Γ-action.[3] The action of Γ is given by

$$\alpha : \begin{cases} X^{\mu}_{I,J}\left(\sigma - 2\pi R_5 \frac{a}{b}\right) = \omega^{I-J} X^{\mu}_{I,J}(\sigma), \quad (\mu = 2,3,4) \\ Q_{I,J}\left(\sigma - 2\pi R_5 \frac{a}{b}\right) = \omega^{1+I-J} Q_{I,J}(\sigma) \\ \tilde{Q}_{I,J}\left(\sigma - 2\pi R_5 \frac{a}{b}\right) = \omega^{-1+I-J} \tilde{Q}_{I,J}(\sigma) \end{cases} \tag{3.1}$$

$$\beta : \begin{cases} X^{\mu}_{I,J}(\sigma + 2\pi R_5) = X^{\mu}_{I,J}(\sigma), \quad (\mu = 2,3,4) \\ Q_{I,J}(\sigma + 2\pi R_5) = Q_{I,J}(\sigma) \\ \tilde{Q}_{I,J}(\sigma + 2\pi R_5) = \tilde{Q}_{I,J}(\sigma) \end{cases} \tag{3.2}$$

where we suppressed the coordinates of the $(2+1)$-dimensional world-volume other than σ, and I, J run from 0 to $q-1$.

For the moment, we focus on the behavior of the field $Q_{I,J}$ under Γ-action. By the projection corresponding to the element $\beta^a \alpha^b$, $Q_{I,J}$ satisfies

$$Q_{I,J}(\sigma) = (\omega^b)^{1+I-J} Q_{I,J}(\sigma)\,. \tag{3.3}$$

From this relation, it follows that unless

$$J \equiv I + 1 \pmod{r}\,, \tag{3.4}$$

Matrix element $Q_{I,J}$ is zero. In the following, we write $I = i + mr$ where $i = 0, \ldots, r-1$ and $m = 0, \ldots, b-1$, and denote the Matrix element $Q_{I,J}$ as $Q^{i,j}_{m,n}$.

[3] In this section, we consider the single $D2$-brane probe ($U(1)$-gauge theory) to avoid unessential complexity. It is straightforward to include further the color indices of $U(N)$, (which means we start from $U(qN)$-gauge theory) but a little too cumbersome for our purpose.

Next we make the projection by the element $\beta^c\alpha^d$ $(ad-bc=1)$:

$$Q^{i,i+1}_{m,n}\left(\sigma+2\pi\frac{R_5}{b}\right)=e^{-i\theta_6(m-n)}Q^{i,i+1}_{m,n}(\sigma)\,,\qquad (i=0,\ldots,r-2)$$
$$Q^{r-1,0}_{m,n}\left(\sigma+2\pi\frac{R_5}{b}\right)=e^{-i\theta_6(m+1-n)}Q^{r-1,0}_{m,n}(\sigma)\,, \tag{3.5}$$

where we set

$$\theta_6=2\pi\frac{d}{b}\,, \tag{3.6}$$

and this is no other than the θ-angle of (p,q) 5-brane gauge theory in the decoupling limit [6, 7]. We can easily find the boundary conditions of the other fields. These conditions for the non-vanishing fields are given by

$$X^{i,i}_{m,n}\left(\sigma+2\pi\frac{R_5}{b}\right)=e^{-i\theta_6(m-n)}X^{i,i}_{m,n}(\sigma)\,,\qquad (i=0,\ldots,r-1)$$
$$\tilde{Q}^{i+1,i}_{m,n}\left(\sigma+2\pi\frac{R_5}{b}\right)=e^{-i\theta_6(m-n)}\tilde{Q}^{i+1,i}_{m,n}(\sigma)\,,\qquad (i=0,\ldots,r-2)$$
$$\tilde{Q}^{0,r-1}_{m,n}\left(\sigma+2\pi\frac{R_5}{b}\right)=e^{-i\theta_6(m-n-1)}\tilde{Q}^{0,r-1}_{m,n}(\sigma)\,. \tag{3.7}$$

Note that the R_5/b-periodicity of the modes which have no monodromies simply means the fractionality of our Matrix string probe. But the existence of monodromies leads to the peculiar behavior of the fractional string. We will later discuss about this point.

One can understand the appearance of θ_6 in the monodromy relation (3.5) (3.7) by the following argument. For this purpose, we replace $X_{p,q}=(\mathbf{S}^1\times\mathbf{C}^2)/\mathbf{Z}_q$ by $Y_{p,q}=(\mathbf{S}^1\times W)/\mathbf{Z}_q$ [6] where W is the charge one Taub-NUT space (whose asymptotic behavior is described by the Hopf fibration). $\mathbf{S}^1\times W$ is fibered over $\mathbf{R}^3$ with fibers $\mathbf{T}^2[\mathbf{e}_5,\mathbf{e}_6]$, where we denote $\mathbf{T}^2[\mathbf{e}_5,\mathbf{e}_6]$ as the quotient of $\mathbf{R}^2$ by the lattice generated by $\mathbf{e}_5=(2\pi R_5,0)$ and $\mathbf{e}_6=(0,2\pi R_6)$, and $\mathbf{S}^1$ fiber of W corresponds to $\mathbf{e}_6$.

The fiber E of $Y_{p,q}=(\mathbf{S}^1\times W)/\mathbf{Z}_q$ is given by $\mathbf{T}^2[\mathbf{e}_5,\mathbf{e}_6]/\mathbf{Z}_q$, where $\mathbf{Z}_q$ acts to the point $\mathbf{x}=x^5\mathbf{e}_5+x^6\mathbf{e}_6$ on $\mathbf{T}^2[\mathbf{e}_5,\mathbf{e}_6]$ by $\mathbf{x}\to\mathbf{x}+(\mathbf{e}_6-p\mathbf{e}_5)/q$. We can see that $E=\mathbf{T}^2[\mathbf{e}_5,\tilde{\mathbf{e}}_6]$ with $\tilde{\mathbf{e}}_6=(\mathbf{e}_6-p\mathbf{e}_5)/q$. The Taub-NUT direction $\mathbf{e}_6$ is represented by the new basis $\{\mathbf{e}_5,\tilde{\mathbf{e}}_6\}$ as $\mathbf{e}_6=p\mathbf{e}_5+q\tilde{\mathbf{e}}_6$. We make $SL(2,\mathbf{Z})$ transformation for the basis of the lattice and take the Taub-NUT direction as one of the basis vector of the lattice, which we denote $\mathbf{a}_6:=a\mathbf{e}_5+b\tilde{\mathbf{e}}_6\equiv\mathbf{e}_6/r$. Then the fiber of $Y_{p,q}$ becomes $E'=\mathbf{T}^2[a\mathbf{e}_5+b\tilde{\mathbf{e}}_6,c\mathbf{e}_5+d\tilde{\mathbf{e}}_6]\equiv\mathbf{T}^2[\mathbf{a}_6,\mathbf{a}_5]$ (outside the singularity). Note that $\mathbf{a}_6$ and $\mathbf{a}_5$ correspond to $\beta^a\alpha^b,\beta^c\alpha^d\in\Gamma$, respectively. Under this $SL(2,\mathbf{Z})$ transformation, a (p,q) 5-brane turns into a $D5$-brane in the Type IIB theory picture. The moduli parameter of E and E' is given by

$$\tau(E)=-\frac{a}{b}+i\frac{R_6}{rbR_5}\,, \tag{3.8}$$
$$\tau(E')=\frac{d}{b}+i\frac{rR_5}{bR_6}\,. \tag{3.9}$$

The decoupling limit $\tau(E) \to i\infty$ is realized by $R_6/R_5 \to \infty$, and in our situation this is always achieved independent of a fixed value of R_5, since we want to take the ALE-limit $R_6 \to \infty$. [4] This fact was first pointed out in [12]. In this limit, $2\pi\tau(E')$ becomes the theta angle $\theta_6 = 2\pi d/b$ of the decoupled 6-dimensional theory.

In the $\mathbf{e}_5, \mathbf{e}_6$ basis, $\mathbf{a}_6, \mathbf{a}_5$ are expressed as $\mathbf{a}_6 = \mathbf{e}_6/r$ and $\mathbf{a}_5 = -\mathbf{e}_5/b + d\mathbf{e}_6/rb$. The monodromy (3.5)(3.7) can be understood from the relation

$$\frac{1}{b}\mathbf{e}_5 = \frac{\theta_6}{2\pi}\mathbf{a}_6 - \mathbf{a}_5\,. \tag{3.10}$$

A fractional string is a membrane which is stretched between a $\mathbf{a}_6$-side of the parallelogram corresponding to $E' = \mathbf{T}^2[\mathbf{a}_6, \mathbf{a}_5]$ and the opposite $\mathbf{a}_6$-side, and extends along the direction $\mathbf{e}_5/b$ which is perpendicular to $\mathbf{a}_6$. (In [5], this configuration of a membrane is called "strip".) From the relation (3.10), when one shifts σ by $2\pi R_5/b$, one does not come back to the same point on the torus but go to the point shifted in the Taub-NUT direction. Because of the α-projection, the fields on a fractional string have the charge under the shift in the Taub-NUT direction. So the shift of σ by $2\pi R_5/b$ leads to the monodromy (3.5), (3.7). This situation is reminiscent of that in [13], in which the noncommutative geometry in Matrix theory[14] is argued. So it may be interesting to discuss the relation between our fractional Matrix string and noncommutative geometry.

In summary, the field theory on the fractional string probe is $(2+1)$-dimensional $U(b)$ A_{r-1} quiver gauge theory on $\mathbf{R}^2 \times \mathbf{S}^1_{R_5/b}$ with the monodromies (3.5) (3.7). [5]

Note that $X^{i,i}$ and $(Q^{i,i+1}, \tilde{Q}^{i+1,i})$ $(0 \le i \le r-2)$ are periodic up to a gauge transformation (with monodromy) , e.g.

$$X^{i,i}(\sigma + 2\pi R_5/b) = UX^{i,i}U^{-1}\,, \tag{3.11}$$

where $U = \mathrm{diag}(1, e^{-i\theta_6}, \ldots, e^{-i(b-1)\theta_6}) \in U(b)$. In other words, the monodromies of these fields can be absorbed into a suitable Wilson line. However, $(Q^{r-1,0}, \tilde{Q}^{0,r-1})$ are *not* periodic up to the same gauge transformation U. This implys that *all* the monodromies of this system *cannot* be replaced with a VEV of Wilson line. It is the essential feature of our .fractional Matrix string theory that there exist monodromies which can never be eliminated.

We will observe later that thanks to this monodromy, the moduli space of our quiver gauge theory behaves in the expected manner for the physics of fractional string.

3.1. Moduli space of the fractional string theory

First, we consider the Higgs branch of the fractional Matrix string, which represents

[4] In [6], the limit $R_5 \to 0$ is rather called the "decoupling limit". But in the stand point of this paper, it is more suitable that the ALE-limit ($R_6 \to \infty$) is called so. In the limit $R_5 \to 0$, the bulk gravity decouples very fast, and hence we shall call it the "strongly decoupling limit".

[5] Of course, if we start from the N D2 probes, we will obtain $U(bN)$ A_{r-1} quiver gauge theory with the similar monodromy as (3.5), (3.7).

the string moving in the ALE direction. The vacuum condition is given by

$$Q^{i,i+1}\tilde{Q}^{i+1,i} - \tilde{Q}^{i,i-1}Q^{i-1,i} = \zeta_i \mathbf{1}_{b\times b}\,. \tag{3.12}$$

We included the Fayet-Iliopoulos parameters ζ_i $(0 \leq i \leq r-1)$ for each $U(1)$ subgroup of gauge group $U(b)_i$ $(0 \leq i \leq r-1)$. These FI-parameters ζ_i should satisfy the relation $\sum_{i=0}^{r-1}\zeta_i = 0$ for the consistency of (3.12). For the quiver gauge theory with the monodromies (3.5) and (3.7), only the components $(Q^{i,i+1}_{n,n}, \tilde{Q}^{i+1,i}_{n,n})$ $(0 \leq i \leq r-2)$ and $(Q^{r-1,0}_{n,n+1}, \tilde{Q}^{0,r-1}_{n+1,n})$ of the hypermultiplets can have the zero-modes and have the vacuum expectation values. We define the gauge invariant variables x, y and z from these components;

$$\begin{aligned} x &= Q^{0,1}_{0,0}\tilde{Q}^{1,0}_{0,0} \\ y &= \prod_{n=0}^{b-1}\left(\prod_{i=0}^{r-2} Q^{i,i+1}_{n,n}\right) Q^{r-1,0}_{n,n+1} \\ z &= \prod_{n=0}^{b-1}\tilde{Q}^{0,r-1}_{n+1,n}\left(\prod_{i=0}^{r-2}\tilde{Q}^{i+1,i}_{n,n}\right)., \end{aligned} \tag{3.13}$$

By the relation (3.12), other gauge invariant variables can be expressed by x;

$$\begin{aligned} Q^{i,i+1}_{n,n}\tilde{Q}^{i+1,i}_{n,n} &= x + a_i \\ Q^{r-1,0}_{n,n+1}\tilde{Q}^{0,r-1}_{n+1,n} &= x + a_{r-1}\,, \end{aligned} \tag{3.14}$$

where $a_0 = 0$ and $a_i = \sum_{k=1}^{i}\zeta_i$, $(i = 1,\ldots,r-1)$. From these relations, the complex structure of the Higgs branch is described by the equation:

$$yz = \left(\prod_{i=0}^{r-1}(x + a_i)\right)^b. \tag{3.15}$$

This is nothing but the equation of the A_{q-1}-ALE space. However, we should remark the following fact: In our quiver theory, only the FI parameters *which are sufficient to resolve the partial* $\mathbf{Z}_r$*-singularity* can be included. So, the $\mathbf{Z}_q/\mathbf{Z}_r \cong \mathbf{Z}_b$-singularity remaines at the every point in the Higgs branch. This gives us the physical picture that only the b joined fractional strings can freely move in the ALE bulk space with the A_{r-1}-singularity.

It may be meaningful to compare this result with that of the usual quiver theory *without* monodromy. This is known [15] to have the structure of a symmetric orbifold; $Sym^b(\mathrm{ALE}(A_{r-1}))$. This fact corresponds to the simple picture that b strings freely move in the A_{r-1}-ALE space, and is not suited to the behavior of fractional string .

Next, let us consider the Coulomb branch $\mathcal{M}_V^{\theta_6}$ of the fractional string theory. This branch should correspond to the fractional string moving inside the (p,q) 5-brane. As in the previous section, the Coulomb branch is parametrized by $\phi^i_m \in \mathbf{R}^3 \times \mathbf{S}^1_{b\tilde{R}_5}$ $(i = 0,\ldots,r-1,\ m = 0,\ldots,b-1)$ (three Cartan components of the vectormultiplet and one Wilson line around $\mathbf{S}^1_{R_5/b}$).

At the tree level, the metric of $\mathcal{M}_V^{\theta_6}$ is diagonal: $ds^{2\ (0)} \propto \delta_{ij}\delta_{mn} d\phi_m^i d\phi_n^j$.

We calculate the metric of $\mathcal{M}_V^{\theta_6}$ to the one-loop order. For the $U(q)$ A_{r-1} quiver gauge theory without monodromy, the metric of the Coulomb branch has the same form as (2.23) with $G_1(\phi_{mn}^{ij}, bl_s^2/R_5)$ replaced by $G_2(\phi_{mn}^{ij}, R_5/b)$ defined as follows;

$$G_2(\phi; R) = \frac{T}{4|\vec{X}|}\left\{1 + 2\sum_{k=1}^{\infty} e^{-m_k|\vec{X}|}\cos(m_k y)\right\}, \tag{3.16}$$

where $\phi = (\vec{X}, y) \in \mathbf{R}^3 \times \mathbf{S}^1_{b\tilde{R}_5}$ (see Appendix for the details). As in the previous case, this coincides with the Green function on $\mathbf{R}^3 \times \mathbf{S}^1_{b\tilde{R}_5}$, and was first introduced in [3].

Now we consider the theory with monodromy. The monodromy changes the mass of particles which run around the loops. Thus the one-loop metric of the fractional string theory can be obtained by replacing ϕ_{mn}^{ij} by the following "modified mass" $\hat{\phi}_{mn}^{ij}$;

$$\begin{aligned} \hat{\phi}_{mn}^{ij} &= \hat{\phi}_m^i - \hat{\phi}_n^j, \qquad (i,j) \neq (r-1,0),(0,r-1) \\ \hat{\phi}_{mn}^{r-1,0} &= -\hat{\phi}_{nm}^{0,r-1} = \hat{\phi}_m^{r-1} - \hat{\phi}_n^0 - (\vec{0}, \theta_6 b\tilde{R}_5)\,, \end{aligned} \tag{3.17}$$

where $\hat{\phi}_m^i = \phi_m^i - (\vec{0}, m\theta_6 b\tilde{R}_5)$. The one-loop metric can be written as

$$ds^{2\ (1)} = \frac{1}{2}\sum_{m,n,i,j} a_{ij}^{mn} G_2(\hat{\phi}_{mn}^{ij}; R_5/b)\left(d\phi_{mn}^{ij}\right)^2. \tag{3.18}$$

Here we again emphasize that the effects of monodromies cannot be obtained by merely shifting the Wilson line, i.e., replacing ϕ_m^i by $\hat{\phi}_m^i$. (This is due to the extra term for $\hat{\phi}_{mn}^{r-1,0}$ in the above expressions (3.17).) This statement corresponds to the fact that all fields in the fractional string theory cannot be made periodic simultaneously by any gauge transformation.

Now, let us argue on the structure of singularity in the Coulomb branch $\mathcal{M}_V^{\theta_6}$. In general, the singularity of the moduli space is the point where the extra massless particle appears. As is mentioned above, the mass of the hypermultiplet is proportional to $\hat{\phi}_{m,m+1}^{i,i+1}$ (3.17).

At the origin of the Coulomb branch $\phi_m^i = 0$, there are extra massless hypermultiplets $(Q_{m,m}^{i,i+1}, \tilde{Q}_{m,m}^{i+1,i})$ $(0 \leq i \leq r-2)$ and $(Q_{m,m+1}^{r-1,0}, \tilde{Q}_{m+1,m}^{0,r-1})$ which have no monodromy. Needless to say, the VEVs of these massless fields parametrizes the Higgs branch above discussed. That is, this branch emanates from the origin of the Coulomb branch.

However, there are other singularities in this branch. One can immediately notice the existence of the next singular points P_j $(j = 0, \ldots, r-1)$ which are distributed $\mathbf{Z}_r$-symmetrically around the origin;

$$P_j: \begin{cases} \phi_m^i = (\vec{0}, m\theta_6 b\tilde{R}_5), & (i = 0, \ldots, j) \\ \phi_m^i = (\vec{0}, (m+1)\theta_6 b\tilde{R}_5), & (i = j+1, \ldots, r-1) \end{cases} \tag{3.19}$$

For example, P_{r-1} is the point where $\hat{\phi}_m^i = 0$. At P_j, the hypermultiplet $(Q^{i,i+1}, \tilde{Q}^{i+1,i})$ $(i \neq j)$ is massless for every $U(b)$ index. (The number of massless particles is much larger than that of the origin!) Nevertheless, the Higgs branch does *not* emanate from this point. In fact, all the components of $(Q^{j,j+1}, \tilde{Q}^{j+1,j})$ are massive, and so, we have no non-trivial solution for the equation of flat direction.

In the same way, we can find many other singular points in the Coulomb branch. But, the point which can make a transition to another branch is only the origin. Only at this point, the b fractional strings can join and generates the Higgs branch, that is, move into the ALE bulk.

For the end of this section, let us consider the asymptotic behavior of the metric of the Coulomb branch in the limit $R_5 \to \infty$ and $R_5 \to 0$. In the limit $R_5 \to \infty$, the world-volume of $D2$-brane $\mathbf{R}^2 \times \mathbf{S}^1_{R_5/b}$ becomes flat three-dimensional space $\mathbf{R}^3$. In this limit, since the effect of monodromy vanishes, i.e. $\hat{\phi}^{ij}_{mn} \to \phi^{ij}_{mn}$, the Coulomb branch $\mathcal{M}^{\theta_6}_V$ is the same as that of the ordinary $U(b)$ A_{r-1} quiver gauge theory on $\mathbf{R}^3$ which is known from the mirror symmery of the $d = 3$ $N = 4$ supersymmetric theory [16, 17] to be the moduli space of b-instantons in $SU(r)$ gauge theory.

On the other hand, in the "strongly decoupling limit" $R_5 \to 0$, the mass of the field with monodromy becomes infinite, so the excitations of these fields decouple. The fields with no monodromies do not decouple, and have the masses; $\hat{\phi}^{ij}_{mm} = \phi^{ij}_{mn}$ $(i, j) \neq (0, r-1), (r-1, 0)$, $\hat{\phi}^{r-1,0}_{m,m+1} = \phi^{r-1,0}_{m,m+1}$. The metric of the Coulomb branch is then reduced to

$$
\begin{aligned}
ds^{2\ (0)} &= \frac{1}{2} \sum_{m} \sum_{\substack{(i,j)\neq(r-1,0)\\(0,r-1)}} \hat{a}_{ij} G_2(\phi^{ij}_{mm}, R_5/b) \Big(d\phi^{ij}_{mm}\Big)^2 + \sum_{m} G_2(\phi^{r-1,0}_{m,m+1}, R_5/b) \Big(d\phi^{r-1,0}_{m,m+1}\Big)^2 \\
&= \frac{1}{2} \sum_{I,J=0}^{q-1} \hat{a}_{IJ} G_2(\phi_{IJ}, R_5/b) \Big(d\phi_{IJ}\Big)^2 , \qquad (3.20)
\end{aligned}
$$

where $\phi_I = \phi^i_m$ with $I = i + mr$ and $\hat{a}_{IJ}$ is the adjacency Matrix of the A_{q-1} affine Dynkin diagram. This metric is the same as that of the $U(1)$ A_{q-1} quiver gauge theory on $\mathbf{R}^2$.

One can understand this phenomenon both from the M-theory and Type IIB pictures. In the M-theory picture, $(\mathbf{S}^1_{R_5} \times \mathbf{C}^2)/\mathbf{Z}_q$ becomes $\mathbf{C}^2/\mathbf{Z}_q$ in the decoupling limit, and so the theory on the fractional string reduces to the usual A_{q-1} quiver theory. In the Type IIB picture, because the tension of the NS5-brane is much larger than that of the $D5$-brane in this limit, a (p, q)-fivebrane becomes effectively q NS5-branes, of which T-dual picture is of course the A_{q-1} ALE.

4. Discussion

We have studied two Matrix string theories as the probe of the (p, q) 5-brane: Witten's Matrix string theory and the fractional string theory. These theories are respectively $(1 + 1)$-dimensional $U(N)$ A_{q-1} quiver gauge theory and $(2 + 1)$-dimensional $U(bN)$ A_{r-1} quiver gauge theory *with monodromy*.

In the Witten's Matrix string theory the monodromy acts as a diagram automorphism (clock-shift) of the extended Dynkin diagram for quiver, which reduces the theory to the A_{r-1} quiver. In this screwing procedure, it naturally incorporate the excitations of fractional string.

On the other hand, the monodromies in the fractional Matrix string theory have different forms — the phase shifts of vector and hypermultiplets, which is discussed in [18]. They play an essential role in the fractional Matrix string. Thanks to them, we can realize the peculiar behavior of fractional string from the viewpoint of Matrix theory. Our analyses of moduli spaces confirm the following expectation; a single fractional string cannot move away from the singular surface (the world-brane of (p, q) 5-brane), and only the joined fractional strings which have the equal tension to a fundamental string can do.

The quiver Matrix theories are "magnetic" (in the usual convention of terminology) formulations for the brane theory probing 5-branes. Thus, it may be an interesting task to construct the fractional Matrix string theory as the "electric" theory. In this framework, the decoupled 6-dimensional physics should be described as Higgs branch, which is the instanton moduli space (this is a tautology, since the fractional string is an instanton from the beginning!), and the joined fractional strings moving away from the (p, q) 5-branes corresponds to the Coulomb branch.

As we mentioned in section 1, this is not an easy problem, since the instanton string is not a D-brane in general. Moreover, in our discussion, so-called the Mirror symmetry for 3-dimensional gauge theory [16, 15] cannot be applicable in the exact sense. This is because, in the case when R_5 is finite, the quantum moduli space of Coulomb branch has no continuous isometry, and so, the electric-magnetic duality is not reduced to the simple "T-dual" transformation [3]. (The case $R_5 = \infty$, the monodromy lose its meaning and our fractional Matrix string reduces to the usual D2-probe.)

Although difficulties exist, we believe it meaningful to construct the electric theory on account of a few reasons. First, for the 5-branes with a *irrational* θ-angles, the ALE description failes. Nevertheless, the decoupled 6-dimensional theory can be similarly defined in the 5-brane framework [7]. This implies that the electric formulation of "instanton Matrix string" (which can have an irrational tension in general) may be also applicable for the irrational cases. Second, let us note the following fact: The configurations of many D5s with non-vanishing θ-angle and many instanton strings are natural generalizations of those corresponding to the Mardacena's $AdS_3 \times S^3$ SCFT [19]. We emphasize that in the cases of non-zero θ-angle, AdS_3 CFT does *not* correspond to the system of D5+D1, *but* to D5+instanton strings. In this meaning, the electric formulation of instanton Matrix string may add a new perspective to the study of AdS_3 CFT.

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A. One-loop integral on $\mathbf{R}^n \times \mathbf{S}^1_R$

One vector multiplet of $(n+1)$-dimensional supersymmetric gauge theory with 8 supercharges contains $(5-n)$ scalar fields, which we denote $\vec{X} \in \mathbf{R}^{5-n}$. The metric of the Coulomb branch is written in terms of the one-loop integral on $\mathbf{R}^n \times \mathbf{S}^1_R$,

$$G_n(\phi; R) = \frac{1}{R} \sum_{k=-\infty}^{\infty} \int \frac{d^n p}{(2\pi)^n} T^2 \left[p^2 + \frac{1}{R^2}(k + TRy)^2 + \left(T|\vec{X}|\right)^2 \right]^{-2}, \tag{A.1}$$

where $y = T^{-1} \int_{\mathbf{S}^1_R} \frac{A}{2\pi R}$ is the Wilson line, which has periodicity $y \sim y + 2\pi\tilde{R}$ with $\tilde{R} = 1/2\pi RT = l_s^2/R$, and $\phi = (\vec{X}, y)$ is the coordinate of $\mathbf{R}^{5-n} \times \mathbf{S}^1_{\tilde{R}}$. After the Poisson resummation, $G_n(\phi; R)$ is rewritten as

$$\begin{aligned} G_n(\phi; R) &= \frac{1}{2}\left(\frac{2\pi}{T^2}\right)^{\nu-1} \sum_{k=-\infty}^{\infty} \left(\frac{|m_k|}{|\vec{X}|}\right)^{\nu} K_\nu(|m_k \vec{X}|) e^{i m_k y} \\ &= \frac{1}{2}\left(\frac{2\pi}{T^2}\right)^{\nu-1} \left\{ \frac{2^{\nu-1}\Gamma(\nu)}{|\vec{X}|^{2\nu}} + 2\sum_{k=1}^{\infty} \left(\frac{m_k}{|\vec{X}|}\right)^{\nu} K_\nu(m_k|\vec{X}|)\cos(m_k y) \right\}. \end{aligned} \tag{A.2}$$

Here we defined $m_k = 2\pi RTk = k/\tilde{R}$ and $\nu = (3-n)/2$. $K_\nu(z)$ is the modified Bessel function;

$$K_\nu(z) = \frac{1}{2}\left(\frac{z}{2}\right)^{\nu} \int_0^\infty dt\, t^{-\nu-1} \exp\left(-t - \frac{z^2}{4t}\right). \tag{A.3}$$

Note that $G_n(\phi; R)$ is the Green function on $\mathbf{R}^{5-n} \times \mathbf{S}^1_{\tilde{R}}$;

$$\left(\Delta_{\vec{X}} + \partial_y^2\right) G_n(\phi; R) = -\frac{T}{2R}\left(\frac{2\pi}{T}\right)^{2\nu} .\delta(\vec{X})\delta(y) \tag{A.4}$$

In the limit $R \to \infty$, $G_n(\phi; R)$ behaves like

$$G_n(\phi; R) \sim \frac{1}{2}\left(\frac{4\pi}{T^2}\right)^{\frac{1-n}{2}} \frac{\Gamma\left(\frac{3-n}{2}\right)}{|\vec{X}|^{3-n}} \tag{A.5}$$

and in the limit $R \to 0$,

$$2\pi R G_n(\phi; R) \sim \frac{1}{2}\left(\frac{4\pi}{T^2}\right)^{\frac{2-n}{2}} \frac{\Gamma\left(\frac{4-n}{2}\right)}{|\phi|^{4-n}}. \tag{A.6}$$

In section 2, we need $G_1(\phi, R)$ which is given by

$$G_1(\phi, R) = \frac{1}{2|\vec{X}|^2} \left\{ 1 + 2\sum_{k=1}^{\infty} m_k |\vec{X}| K_1\left(m_k|\vec{X}|\right) \cos(m_k y) \right\} \tag{A.7}$$

and $G_2(\phi, R)$ is relevant for the fractional string theory,

$$G_2(\phi; R) = \frac{T}{4|\vec{X}|} \left\{ 1 + 2\sum_{k=1}^{\infty} e^{-m_k|\vec{X}|} \cos(m_k y) \right\}, \tag{A.8}$$

where we used the fact that $K_{1/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z}$.

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